OPTIMAL INVESTMENT WITH STOCKS AND DERIVATIVES

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Abstract. This paper studies the problem of maximizing expected utility from terminal wealth, combining a static position in derivative securities with a traditional dynamic trading strategy in stocks. We work in the framework of a general semi-martingale model and consider a utility function defined on the positive real line.

1 Introduction

A classical problem in financial economics is that of an investor who wants to maximize his expected utility from terminal wealth by investing in a frictionless market. This problem was first studied in continuous time by Merton [Mer69], [Mer71], using dynamic programming arguments.

The introduction by Harrison and Kreps [HK79], Harrison and Plinska [HP81] and Ross [Ros76] of the notion of equivalent martingale measure made it possible to overcome the restriction of a Markovian framework through the use of convex duality methods. Many important papers that followed this approach have been published, as the problem was solved in increased generality (most notable was the extension to incomplete markets, see e.g. [IKX91]); we refer to the articles by Kramkov and Schachermayer [KS99],[KS03] for a survey.

In this paper we solve the analogous problem of maximizing expected utility from terminal wealth, in a market where some of the contingent claims can be traded *only* at time zero. We will think of the liquid part of the market as being composed of stocks, and the illiquid one by derivatives.

This problem, has been considered in Ilhan, Jonsson and Sircar [AIS05], who work in the framework of an exponential utility and of a general semi-martingale model. They make essential use of relative entropy techniques and of some explicit representations of the maximal utility and of indifference prices; thus, as they point out, the 'extension to more general cases is not trivial', as a different approach is needed. Notice moreover how this problem does not follow under the general umbrella of utility maximization with convex constraints (see [LZ] for a survey), since it cannot be re-phrased asking that the portfolio and wealth process lie in some given convex set (possibly depending on t and ω); rather, we demand that

the investor, after choosing his position at time zero *arbitrarily*, keep his position in derivatives unchanged for the rest of the time horizon, while freely investing in stocks.

To our knowledge and perhaps surprisingly, nobody has yet treated this problem in the case of a general utility function defined on the positive real line. This we do, in the setting of the general semi-martingale model employed in Hugonnier and Kramkov [HK04], using convex duality methods; our first main theorem is an analogue of the results found in Kramkov and Schachermayer [KS99],[KS03], on which we rely. Notice that our problem can be decomposed in two steps: choosing the optimal amount of derivatives to buy at time zero, and then invest optimally in the continuous time stock market. The second step is the problem of optimal investment with random endowment, which is then closely related to our problem; our second main theorem describes this relationship.

The rest of the paper is organized as follows. In Section 2 we present the model of financial market and we define our problem. In Section 3 we state our main results, and in Section 4 we prove the first one, of which in Section 5 we provide some variations. In Section 6 we generalize some results of Hugonnier and Kramkov [HK04], so that in Section 7 we can prove our second main result.

2 The model

Consider at first a model of a financial market composed of a savings account and d stocks which can be traded continuously in time. As is common in mathematical finance, we consider a finite time horizon $[0, T]$, and we assume that the interest rate is 0; that is, the price process of the savings account is used as numéraire and is thus normalized to one. The price process $S = (S^i)_{i=1}^d$ of the stocks is assumed to be a semi-martingale on a given filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$, where the filtration $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$ satisfies the usual conditions.

Now enlarge the market by allowing also n contingent claims $f = (f_j)_{j=1}^n$ to be traded at price $p = (p_j)_{j=1}^n$; we assume that these contingent claims can be traded *only* at time zero, and that p is an arbitrage-free price for the European contingent claims f (in a sense which will be specified later).

A self-financing portfolio is then defined as a triple (x, q, H), where $x \in \mathbb{R}$ represents the initial capital, $q_j \in \mathbb{R}$ represents the holding in the contingent claim f_j, and the random variable H_t^i specifies the number of shares of stock i held in the portfolio at time t.

An agent with portfolio (x, q, H) will invest his initial wealth x buying q European contingent claims at price p at time zero. This quantity is then held constant up to maturity, so the vector q represents the illiquid part of the portfolio and $qp := \sum_{j=1}^n q_j p_j$ represents the wealth invested in the European contingent claims (in this paper vw will always denote the dot product between two vectors v and w, and $|\cdot|$ will denote the Euclidean norm in $\mathbb{R}^{n+1}$).

He will then invest the remaining wealth $x - qp$ dynamically, buying H_t share

of stocks at time $t \in [0,T]$, and put the rest (positive or negative) into the savings account. We will denote by X_t the value of the dynamic part of the portfolio, which will be called simply the wealth process; $x - qp$ will be called the initial value of the wealth process (which is different from the initial wealth x of the portfolio). The wealth process X evolves in time as the stochastic integral of H with respect to S:

$$X_t = x - qp + (H \cdot S)_t = x - qp + \int_0^t H_u dS_u, \qquad t \in [0,T],$$

where H is assumed to be a predictable S-integrable process.

For $x \geq 0$, we denote by $\mathcal{X}(x)$ the set of non-negative wealth processes whose initial value is equal to x, that is,

$$\mathcal{X}(x) := \{X \geq 0 : X_t = x + (H \cdot S)_t\}.$$

A probability measure Q is called an equivalent local martingale measure if it is equivalent to P and if every $X \in \mathcal{X}(1)$ is a local martingale under Q.[1]

We denote by $\mathcal{M}$ the family of equivalent local martingale measures, and we assume that

$$\mathcal{M} \neq \emptyset. \tag{1}$$

This condition is essentially equivalent to the absence of arbitrage opportunities in the market without the European contingent claims: see Delbaen and Schachermayer [DS94] and [DS98] for precise statements as well as for further references.

In our model we consider an economic agent whose preferences over terminal consumption bundles are represented by a utility function $U : (0,\infty) \to \mathbb{R}$. The function U is assumed to be strictly concave, strictly increasing and continuously differentiable and to satisfy the Inada conditions:

$$U'(0) := \lim_{x \to 0+} U(x) = \infty, \qquad U'(\infty) := \lim_{x \to \infty} U'(x) = 0. \tag{2}$$

It will be convenient to consider U defined on the whole real line. We want its extension to be concave and upper semi-continuous, and (2) implies that there is only one possible choice: we define $U(x)$ to be $-\infty$ for x in $(-\infty, 0)$, and to equal $U(0+) := \lim_{x \to 0} U(x)$ at $x = 0$.

We denote by $f = (f_j)_{j=1}^n$ the family of the $\mathcal{F}_T$-measurable payment functions of the European contingent claims with maturity T, and by $qf = \sum_{j=1}^n q_j f_j$ the payoff of the static part of the portfolio. The total payoff of the portfolio (x,q,H) is then $x - qp + (H \cdot S)_T + qf$.

Following [HK04], we assume that the European contingent claims are dominated by the final value of a non-negative wealth process X', that is,

$$|f| := \sqrt{\textstyle\sum_{j=1}^n f_j^2} \leq X'_T\,. \tag{3}$$

[1] We remark that the results in [KS99],[KS03],[HK04],[DS97] on which our proofs hinge, although proved for a locally bounded semi-martingale and with a definition of martingale measure specialized to that case, are true also without the local boundedness assumption if one takes the previous definition of martingale measure.

A non-negative wealth process in $\mathcal{X}(x)$ is said to be maximal if its terminal value cannot be dominated by that of any other process in $\mathcal{X}(x)$. We recall that the set of positive maximal wealth processes coincides with the set of 'good' numéraires in the model (see [DS95]). Since given a wealth process $X \in \mathcal{X}(x)$ one can always find a maximal wealth process $\tilde{X} \in \mathcal{X}(x)$ such that $X_T \leq \tilde{X}_T$ (see [DS94, Proof of Lemma 4.3]), we can and will assume without loss of generality that the process X' that appears in (3) is maximal, so that condition (3) is equivalent to saying that $|f|$ is bounded with respect to some numéraire.

In the classical problem of optimal investment of stocks with a utility function defined on the positive real line, $\mathcal{X}(x)$ constitutes the optimization set. In the presence of additional contingent claims, one has to extend the domain of the problem and to consider wealth processes with possibly negative values. If the contingent claims are uniformly bounded, as in [JCW01], then the optimization set coincides with the set of admissible strategies, i.e., those whose wealth processes are uniformly bounded from below. If the contingent claims are bounded just with respect to some numéraire, the optimization set has to be extended analogously.

We say that a wealth process X is *acceptable* if it admits a representation of the form $X = X' - X''$, where X' is a non-negative wealth process and X'' is a maximal wealth process.[2] Note that if the maximal process $1 + X''$ is chosen as a numéraire, then the discounted process $X/(1+X'')$ is uniformly bounded from below and hence is admissible under this numéraire. Vice versa if X is admissible under a numéraire N, i.e., if X/N is uniformly bounded below by some negative constant c for some positive maximal wealth process N, then the representation $X = X' - X''$ with $X' = X - cN, X'' = -cN$, shows that X is acceptable. Thus the acceptable strategies are the ones that are admissible under some numéraire, and constitute the natural optimization set when the contingent claims f are assumed to satisfy (3).

The objective of this paper is to study the problem of utility maximization in the enlarged market consisting of the bond, the stocks and the contingent claims, i.e., the following optimization problem[3]

$$\tilde{u}(x) := \sup \{\mathbb{E}[U(X_T + qf)] : X \text{ is acceptable }, q \in \mathbb{R}^n,\ X_0 = x - qp\}. \tag{4}$$

Clearly, if the wealth process does not satisfy $X_T + qf \geq 0$ then the corresponding expected utility will be $-\infty$. With this in mind and following [HK04], we define $\mathcal{X}(x, q)$ to be the set of acceptable wealth processes with initial value x whose terminal value dominates the random payoff $-qf$, i.e.,

$$\mathcal{X}(x, q) := \{X : X \text{ is acceptable}, X_0 = x \text{ and } X_T + qf \geq 0\}.$$

In the case when $x \geq 0$ and $q = 0$, this set coincides with the set of non-negative wealth processes with initial value x; in other words, we have $\mathcal{X}(x, 0) = \mathcal{X}(x)$ for all

[2] For a detailed discussion of maximal and acceptable processes, we refer the reader to [DS97].

[3] By convention, we set $\mathbb{E}[U(X_T + qf)]$ equal to $-\infty$ when $\mathbb{E}[U^-(X_T + qf)] = -\infty$ (whether or not $\mathbb{E}[U^+(X_T + qf)]$ is finite).

$x \geq 0$. We will call $\bar{\mathcal{K}}$ the set of points (x,q) where $\mathcal{X}(x,q)$ is not empty, i.e.,

$$\bar{\mathcal{K}} := \{(x,q) \in \mathbb{R} \times \mathbb{R}^n : \mathcal{X}(x,q) \neq \emptyset\}. \tag{5}$$

Consider the problem of utility maximization in presence of a random endowment[3,4]

$$u(x,q) := \sup_{X \in \mathcal{X}(x,q)} \mathbb{E}[U(X_T + qf)], \quad (x,q) \in \mathbb{R}^{n+1}. \tag{6}$$

Then obviously

$$\tilde{u}(x) = \sup_{q \in \mathbb{R}^n} u(x - qp, q) = \sup_{q \in \mathbb{R}^n : (x-qp,q) \in \bar{\mathcal{K}}} u(x - qp, q). \tag{7}$$

3 Statement of the main theorems

To state the main theorems we need to introduce some notation. Denote by $\mathcal{Y}(y)$ the family of non-negative processes Y with initial value y and such that for any non-negative wealth process X the product XY is a super-martingale, that is,

$$\mathcal{Y}(y) := \{Y \geq 0 : Y_0 = y, XY \text{ is a super-martingale for all } X \in \mathcal{X}(1)\}.$$

In particular, as $\mathcal{X}(1)$ contains the constant process 1, the elements of $\mathcal{Y}(y)$ are non-negative super-martingales. Note also that the set $\mathcal{Y}(1)$ contains the density processes of all $Q \in \mathcal{M}$. Given an arbitrary vector $(y,r) \in \mathbb{R} \times \mathbb{R}^n$, we denote by $\mathcal{Y}(y,r)$ the set of non-negative super-martingales $Y \in \mathcal{Y}(y)$ such that the inequality

$$\mathbb{E}[Y_T(X_T + qf)] \leq xy + qr$$

holds whenever $(x,q) \in \bar{\mathcal{K}}$ and $X \in \mathcal{X}(x,q)$. Of course this set will be empty for some values of (y,r).

The convex conjugate function V of the agent's utility function U is defined to be the Fenchel-Legendre transform of the function $-U(-\cdot)$; that is,

$$V(y) := \sup\nolimits_{x>0}(U(x) - xy), \quad y > 0.$$

It is well known that, under the Inada conditions (2), the conjugate of U is a continuously differentiable, strictly decreasing and strictly convex function satisfying $V'(0) = -\infty$, $V'(\infty) = 0$ and $V(0) = U(\infty)$, $V(\infty) = U(0)$ as well as the following bi-dual relation:

$$U(x) = \inf\nolimits_{y>0}(V(y) + xy), \quad x > 0.$$

We now define the problems dual to (4) and to (6) to as follows[4]:

$$\tilde{v}(y) = \inf_{Y \in \mathcal{Y}(y,yp)} \mathbb{E}[V(Y_T)], \quad y \in \mathbb{R} \tag{8}$$

[4]We use the convention that the sup (inf) over an empty set takes the value $-\infty$ $(+\infty)$.

and

$$v(y,r) = \inf_{Y \in \mathcal{Y}(y,r)} \mathbb{E}[V(Y_T)], \quad (y,r) \in \mathbb{R} \times \mathbb{R}^n, \tag{9}$$

where p is the vector of prices of the contingent claims f.

Following [HK04], we will denote by w the value function of the problem of optimal investment without the European contingent claims, and by $\tilde{w}$ its dual value function. In other words

$$w(x) := \sup_{X \in \mathcal{X}(x)} \mathbb{E}[U(X_T)], \quad x > 0; \qquad \tilde{w}(y) := \inf_{Y \in \mathcal{Y}(y)} \mathbb{E}[V(Y_T)], \quad y > 0.$$

Recall that a random variable g is said to be *replicable* if there is an acceptable process X such that $-X$ is also acceptable and $X_T = g$. Provided that it exists, such a process X is unique and is called the replication process for g.

We will say that p is an *arbitrage-free price* for the European contingent claims f if any portfolio with zero initial capital and non-negative final wealth has identically zero final wealth; in other words p is an arbitrage-free price if, for any $q \in \mathbb{R}^n$, $X \in \mathcal{X}(-pq, q)$ implies $X_T = -qf$.

At times we will also need to assume that

$$\text{for any non-zero } q \in \mathbb{R}^n \text{ the random variable } qf \text{ is not replicable.} \tag{10}$$

Notice that, by discarding some contingent claims, we can always assume that (10) holds (see [HK04, Remark 6]). The following is our first main theorem; it is an analogue of the results found in [KS03].

Theorem 1 *Assume that p is an arbitrage-free price for f, that conditions (1), (2) and (3) hold, and that*

$$\tilde{w}(y) < \infty \text{ for all } y > 0.$$

Then one has:

1. *The value functions $\tilde{u}$ and $-\tilde{v}$ are finite, continuously differentiable, strictly increasing and strictly concave on $(0, \infty)$ and satisfy:*

$$\tilde{u}'(0) := \lim_{x \to 0} \tilde{u}'(x) = \infty, \qquad \tilde{v}'(\infty) := \lim_{y \to \infty} \tilde{v}'(y) = 0,$$

$$\tilde{u}'(\infty) := \lim_{x \to \infty} \tilde{u}'(x) = 0, \qquad \tilde{v}'(0) := \lim_{y \to 0} \tilde{v}'(y) = -\infty,$$

as well as the bi-conjugacy relationships:

$$\tilde{u}(x) = \min_{y>0} (\tilde{v}(y) + xy), \qquad x > 0,$$
$$\tilde{v}(y) = \max_{x>0} (\tilde{u}(x) - xy), \qquad y > 0$$

where the unique optimizers are given by $\tilde{y} = \tilde{u}'(x)$ and $x = -\tilde{v}'(\tilde{y})$.

2. $\mathcal{Y}(y, yp) \neq \emptyset$ *and, for any* $y > 0$, *the solution* $\tilde{Y}(y)$ *to (8) exists and is unique.*

3. *The solution* $(\tilde{X}(x), \tilde{q}(x))$ *to (4) exists for any* $x > 0$, *and* $-\tilde{X}(x)$ *is an acceptable wealth process. The final payoff* $\tilde{X}_T(x) + \tilde{q}(x)f$ *is unique.*

4. *If* $\tilde{y} = \tilde{u}'(x)$ *then the optimizers of (4) and (8) satisfy*

$$\tilde{Y}_T(\tilde{y}) = U'(\tilde{X}_T(x) + \tilde{q}(x)f)\,,$$
$$\mathbb{E}[\tilde{Y}_T(\tilde{y})(\tilde{X}_T(x) + \tilde{q}(x)f)] = x\tilde{y}\,.$$

The relationship between problems (4), (6), and (7) is elucidated in the following theorem, our second main result.

Theorem 2 *Under the assumptions of Theorem 1, the following holds:*

1. *For any* $x > 0$ *the solution* $(x - \tilde{q}p, \tilde{q})$ *to (7) exists, and is given by* $-\partial v(\tilde{y}, \tilde{y}p)$, *where* $\tilde{y} = \tilde{u}'(x)$.

2. *For every* $(x, q) \in \{u > -\infty\}$, *the solution* $X(x, q)$ *to (6) exists and is unique.*

3. *The solutions* $(\tilde{X}(x), \tilde{q}(x))$ *to (4) are given by*

$$\{(X(x - qp, q), q) : (x - qp, q) \text{ solves (7)}\}.$$

4. *The following conditions are equivalent:*

   (a) *The solution to (4) is unique for all* $x > 0$.

   (b) *The solution to (7) is unique for all* $x > 0$.

   (c) *Condition (10) holds.*

   (d) *The function* v *is differentiable on* $\mathcal{L}$.

   If these conditions hold, the solution to (7) is given by $(x - \tilde{q}p, \tilde{q}) = -\nabla v(\tilde{y}, \tilde{y}p)$, *where* $\tilde{y} = \tilde{u}'(x)$.

Notice that Theorem 2 allows to compute $\tilde{q}(x)$ explicitly, as long as one can compute v; results on how to approximate $\tilde{q}(x)$ are contained in [KS06].

In Theorem 2, the delicate point is that in general problem (7) does not have a solution if we were to replace $\bar{\mathcal{K}}$ with its interior[5]; however, in the existing literature problem (6) has been solved only in the case where (x, q) belongs to the interior $\mathcal{K}$ of $\bar{\mathcal{K}}$. As a consequence, to compare the problems (4) and (7) and establish Theorem 2, we need to extend the results contained in Hugonnier and Kramkov [HK04].

Of course, problem (4) could be tackled by solving first (7), and then (6); in other words, we could use Theorem 2 to prove (parts of) Theorem 1. However,

[5]For an explicit example where the maximizer belongs to the boundary of $\bar{\mathcal{K}}$, see [Sio12a, Section 4]).

this approach falls short of identifying the dual problem, and in particular does not yield item 4 of Theorem 1. This is why we choose to take another route: we reduce problem (4) to an abstract setting where we can apply the results of Kramkov and Schachermayer [KS99], [KS03]; this has the additional benefit of providing versions of Theorem 1 which hold under a different set of hypotheses.

Notice that the utility function $\tilde{u}$ and its maximizer $(\tilde{q}, \tilde{X}_T)$ depend on the price p of the contingent claims f. In the companion article [Sio12b] we investigate the nature of this dependence, which is a feature that does not have an analogue in [KS03].

4 Optimal investment with stocks and derivatives

To prove our main theorem, we want to apply [KS03, Theorem 4]; in order to do that, we need some preparation.

As shown in [HK04], assumption (3) implies that the convex cone $\bar{\mathcal{K}}$ defined in (5) is closed and its interior $\mathcal{K}$ contains $(x, 0)$ for any $x > 0$, so $\bar{\mathcal{K}}$ is the closure of $\mathcal{K}$. If we define the set $\bar{\mathcal{L}}$ to be the polar of $-\bar{\mathcal{K}}$:

$$\bar{\mathcal{L}} := -\bar{\mathcal{K}}^o := \{v \in \mathbb{R}^{n+1} : vw \geq 0 \text{ for all } w \in \bar{\mathcal{K}}\},$$

then clearly $\bar{\mathcal{L}}$ is a closed convex cone. We will denote by $\mathcal{L}$ its relative interior, so $\bar{\mathcal{L}}$ is the closure of $\mathcal{L}$.

Define the following sets:

$$\mathcal{C}(x, q) := \{g \in L^0_+ : g \leq X_T + qf \text{ for some } X \in \mathcal{X}(x, q)\},$$

and

$$\tilde{\mathcal{C}}(x) := \bigcup_{q \in \mathbb{R}^n} \mathcal{C}(x - qp, q) = \bigcup_{q \in \mathbb{R}^n : (x - qp, q) \in \bar{\mathcal{K}}} \mathcal{C}(x - qp, q). \tag{11}$$

We will write $\tilde{\mathcal{C}}$ as a shorthand for $\tilde{\mathcal{C}}(1)$, and we observe that $x\tilde{\mathcal{C}} = \tilde{\mathcal{C}}(x)$.

Define $\mathcal{D}(y, r)$ to be the set of positive random variables dominated by the final value of some element of $\mathcal{Y}(y, r)$, i.e.,

$$\mathcal{D}(y, r) := \{h \in L^0_+ : h \leq Y_T \text{ for some } Y \in \mathcal{Y}(y, r)\} \tag{12}$$

We will write $\tilde{\mathcal{D}}$ as a shorthand for $\mathcal{D}(1, p)$, and we observe that $\mathcal{D}(y, r) \neq \emptyset$ if and only if $(y, r) \in \bar{\mathcal{L}}$ (see [Sio12a, Remark 5]), and $y\tilde{\mathcal{D}} = \mathcal{D}(y, yp)$.

We recall here two facts proved in [HK04, Lemmas 8 and 9]. Let $\mathcal{M}'$ be the set of equivalent local martingale measures Q such that the maximal process X' that appears in (3) is a uniformly integrable martingale under Q, and let $\mathcal{M}'(p)$ be the subset of measures $Q \in \mathcal{M}'$ such that $\mathbb{E}_Q[f] = p$. If $(1, p) \in \mathcal{L}$ and conditions (1) and (3) hold, then

$$\mathcal{M}'(p) \neq \emptyset, \text{ and if } Q \in \mathcal{M}'(p) \text{ then } dQ/dP \in \mathcal{D}(1, p). \tag{13}$$

We are now ready to prove the analogue of [KS99, Proposition 3.1]

Theorem 3 *Assume that p is an arbitrage-free price for f and that conditions (1), (3), (10) hold. Then $\tilde{\mathcal{C}}$ is bounded in $L^0(\Omega, \mathcal{F}, P)$ and it contains the constant function $g = 1$. The sets $\tilde{\mathcal{C}}$ and $\tilde{\mathcal{D}}$ satisfy the bipolar relations:*

$$g \in \tilde{\mathcal{C}} \iff g \in L^0_+ \text{ and } \mathbb{E}[gh] \leq 1 \quad \forall h \in \tilde{\mathcal{D}} \tag{14}$$

$$h \in \tilde{\mathcal{D}} \iff h \in L^0_+ \text{ and } \mathbb{E}[gh] \leq 1 \quad \forall g \in \tilde{\mathcal{C}}\,. \tag{15}$$

PROOF OF THEOREM 3. The implication $\Rightarrow$ in (14) and in (15), and the inclusion $1 \in \tilde{\mathcal{C}}$ follow directly from definitions. Now we use (13). If $g \in \tilde{\mathcal{C}}$ and $Q \in \mathcal{M}'(p)$ then $\mathbb{E}_Q[g] \leq 1$; it follows that $\tilde{\mathcal{C}}$ is bounded in $L^1(Q)$, and so also in $L^0(P)$, since Q is equivalent to P.

To finish the proof of (15) assume that h is a non-negative random variable such that $\mathbb{E}[gh] \leq 1 \quad \forall g \in \tilde{\mathcal{C}}$. Then, in particular, $\mathbb{E}[X_T h] \leq 1 \quad \forall X \in \mathcal{X}(1)$, so [KS99, Proposition 3.1] implies the existence of a process $Y \in \mathcal{Y}(1)$ such that $h \leq Y_T$. Define the process Z by setting

$$Z_t := \begin{cases} Y_t & \text{if } t < T \\ h & \text{if } t = T. \end{cases}$$

Then Z belongs to $\mathcal{Y}(1)$ and so, since $\mathcal{C}(x, q) \subseteq (x + qp)\tilde{\mathcal{C}}$, it belongs to $\mathcal{Y}(1, p)$. This proves $h \in \tilde{\mathcal{D}}$.

To conclude, let us prove that $\tilde{\mathcal{C}}$ is closed with respect to the convergence in measure; the version of the bipolar theorem found in [BS99] then yields (14). So take $g_n \in \mathcal{C}(1 - pq_n, q_n)$ and assume without loss of generality that g_n converges almost surely to g, and let's prove that $g \in \tilde{\mathcal{C}}$. Since [Sio12a, Lemma 3] implies that $(1 - pq_n, q_n)$ is bounded, passing to a subsequence we can assume that q_n is converging to some q, so [Sio12a, Lemma 4] shows that $g \in \tilde{\mathcal{C}}$ $\square$

We will need the following simple remark, which follows from the fact that a wealth process X is maximal iff there is a measure $Q \in \mathcal{M}$ such that X is a $Q-$uniformly integrable martingale (see [DS97, Theorem 2.5]).

Remark 4 *Any acceptable wealth process with zero initial value and non-negative terminal wealth is indistinguishable from zero, i.e., $\mathcal{X}(0, 0) = \{0\}$*

PROOF OF THEOREM 1. As explained in [HK04, Remark 6], we can assume without loss of generality that (10) holds. Since clearly we have

$$\tilde{u}(x) = \sup\nolimits_{g \in \tilde{\mathcal{C}}} \mathbb{E}[U(xg)], \qquad \tilde{v}(y) = \inf\nolimits_{h \in \tilde{\mathcal{D}}} \mathbb{E}[V(yh)],$$

Theorem 3 puts us in a position to apply [KS03, Theorem 4], as long as we prove that $\tilde{v}$ is finite at all points under the assumption that $\tilde{w}$ is; this follows from [Sio12a, Lemma 3] and [HK04, Lemma 2].

To prove that $-\tilde{X}(x)$ is acceptable, observe that there exists a process $X \in \mathcal{X}(x - \tilde{q}(x)p, \tilde{q}(x))$ such that $-X$ is acceptable and $X_T \geq \tilde{X}_T(x)$ (see [Sio12a, Lemma 2]); since $\tilde{X}_T(x)$ is an optimizer, this implies $X_T = \tilde{X}_T(x)$. Then $\tilde{X}(x) - X \in \mathcal{X}(0, 0)$, so Remark 4 gives the thesis.

5 Alternative statements

It trivially follows from [KS99, Theorem 3.2], that a convenient sufficient condition for the validity of the hypotheses of Theorem 1 is that the asymptotic elasticity of U is strictly less than one.

Analogously, we can rely on [KS99, Theorem 3.1] to get a weaker version of Theorem 1 in the case where we do not assume that $\tilde{w}$ is finite, but only that $\tilde{u}(x) < \infty$ for some $x > 0$. The next remark shows that this assumption is equivalent to the following more natural and apparently weaker condition (16) (thus also to $\tilde{u}$ being finite on $\mathbb{R}_+$).

Remark 5 *Assume that p is an arbitrage-free price for f, that conditions (1), (2), (3) hold and that*

$$w(x) < \infty \text{ for some } x > 0. \tag{16}$$

Then $\tilde{u}(x) < \infty$ for all $x > 0$.

PROOF. As explained in [HK04, Remark 6], we can assume without loss of generality that (10) holds. Let u be the function defined in (6), so that $u(x, 0) = w(x) < \infty$. Then, since u is concave and $u > -\infty$ on $\mathcal{K}$, u never takes the value ∞, so $-u$ is a proper convex function. It follows that u is bounded above by an affine function. Since [Sio12a, Lemma 3] implies that, for any $x > 0$, $\bar{\mathcal{K}}^p(x)$ is bounded, (7) implies $\tilde{u}(x) < \infty$ for all $x > 0$. □

6 Optimal investment with random endowment

We will denote by $Im(\partial f)$ the image of the sub-differential of a concave function $f : \mathbb{R}^{n+1} \to (-\infty, \infty]$ (or of a convex and $[-\infty, \infty)$ valued function), by $dom(\partial f)$ its domain $\{z : \partial f(z) \neq \emptyset\}$, and by

$$\frac{\partial^+ f}{\partial w}(z) := \lim_{t \to 0+} \frac{f(z + tw) - f(z)}{t}$$

the right sided directional derivative of f at z in the direction of w.

The following generalizes [HK04, Theorem 2] by considering also the behavior on the boundary of $\mathcal{K}$ and $\mathcal{L}$.

Theorem 6 *Assume that conditions (1), (3), (2) hold, and that*

$$\tilde{w}(y) < \infty \text{ for all } y > 0,$$

Then the following holds:

1. *The functions u and $-v$ defined on $\mathbb{R}^{n+1}$ have values in $[-\infty, \infty)$, are concave and upper semi-continuous and satisfy the bi-conjugacy relationships:*

$$u(x, r) = \inf_{(y,r) \in \mathbb{R}^{n+1}} (v(y, r) + xy + qr), \qquad (x, q) \in \mathbb{R}^{n+1}, \tag{17}$$

$$v(y, r) = \sup_{(x,q) \in \mathbb{R}^{n+1}} (u(x, r) - xy - qr), \qquad (y, r) \in \mathbb{R}^{n+1}, \tag{18}$$

2. *$\mathcal{K} \subseteq dom(\partial u) \subseteq \{u > -\infty\} \subseteq \bar{\mathcal{K}}$ and all inclusions can be strict.*

3. *$\mathcal{L} = dom(\partial v) \subseteq \{v < \infty\} \subseteq \bar{\mathcal{L}}$. In particular $\partial v(\mathcal{L}) = dom(\partial u)$ and $\mathcal{L} = \partial u(\bar{\mathcal{K}})$ and if (y,r) belongs to the relative boundary of $\mathcal{L}$ then $\partial v(y,r) = \emptyset$ and $\frac{\partial^+ v}{\partial w}(y,r) = -\infty$ for every $w \in \mathcal{L} - (y,r)$.*

4. *For all $(x,q) \in \{u > -\infty\}$ there exists a unique maximizer $X(x,q)$ of (6) and for all $(y,r) \in \{v < \infty\}$ there exists a unique minimizer $Y(y,r)$ of (9). Moreover, $-X(x,q)$ is an acceptable wealth process.*

5. *If $(y,r) \in \partial u(x,q)$, then the terminal values of the optimizers are related by*

$$Y_T(y,r) = U'(X_T(x,q) + qf), \tag{19}$$

$$\mathbb{E}[Y_T(y,r)(X_T(x,q) + qf)] = xy + qr. \tag{20}$$

PROOF. Since u is concave and takes real values on the open set $\mathcal{K}$ (see [HK04, Lemma 2, Theorem 2]), u never takes the value ∞ and the inclusions in item 2 hold. Analogously, since v is convex and takes real values on the open set $\mathcal{L}$ (see [HK04, Lemma 2]), v never takes the value $-\infty$ and the chain of inclusions in item 3 holds (however $dom(\partial v) \subseteq \mathcal{L}$ still needs to be proved).

By [Sio12a, Theorem 6], u is upper-semi-continuous, and there exists a unique maximizer of (6) for any $(x,q) \in \{u > -\infty\}$. To prove that v is lower semi-continuous and that there exists a unique minimizer for any $(y,r) \in \{v < \infty\}$ let $h_n \in \mathcal{D}(y_n, r_n)$ for some converging sequence (y_n, r_n), and define $s := \sup_n y_n < \infty$. Then $h_n \in \mathcal{D}(s)$, so [KS99, Lemma 3.2] gives that the sequence $(V^-(h_n))_{n \geq 1}$ is uniformly integrable. The proof of [Sio12a, Theorem 3] then applies, mutatis mutandis, completing the proof of item 4.

To prove the bi-conjugacy relationships (17) and (18), call $\bar{u}$ the function defined by the right-hand side of (17). Since $v = \infty$ outside $\bar{\mathcal{L}}$, the infimum defining $\bar{u}$ can equivalently be taken over $\bar{\mathcal{L}}$ instead of $\mathbb{R}^{n+1}$, and [Roc70, Theorem 7.5] implies that we can equivalently replace $\bar{\mathcal{L}}$ with $\mathcal{L}$ (analogously we can replace $\mathbb{R}^{n+1}$ with $\bar{\mathcal{K}}$ or $\mathcal{K}$ in (18)). Then u and $\bar{u}$, which are defined on $\mathbb{R}^{n+1}$, never take the value ∞, coincide on $\mathcal{K}$ (as shown in [HK04, Theorem 2]) and so on $\bar{\mathcal{K}}$[6]. By definition u is identically $-\infty$ outside $\bar{\mathcal{K}}$; let us show that this is also true of $\bar{u}$, so they coincide everywhere. If $(x,q) \notin \bar{\mathcal{K}}$ one can find $(y,r) \in \mathcal{L}$ such that $xy + qr < 0$, and so

$$\frac{\bar{u}(x,q)}{n} \leq \left(\frac{v(n(y,r)) + n(xy + qr)}{n} \right).$$

The thesis then follows taking limits for $n \to \infty$, using l'Hospital rule and the fact that, by Theorem 1, the function $\tilde{v}(\lambda) := v(\lambda(1, r/y))$ satisfies $\lim_{\lambda \to \infty} \tilde{v}'(\lambda) = 0$ (r/y is an arbitrage-free price: see [Sio12a, Lemma 3]). This concludes the proof of (17), and now (18) follows from the fact that v is convex, proper and lower semi-continuous, concluding the proof of item 1.

[6]Since they are concave and upper semi-continuous, we can apply again [Roc70, Theorem 7.5])

Let us prove item 5. If $(y,r) \in \partial u(x,q)$ then $(x,q) \in \partial v(y,r)$ and so, by definition of sub-differential, $(x,q) \in \{u > -\infty\}$ and $(y,r) \in \{v < \infty\}$. Item 4 implies the existence of the optimizers $X(x,q)$ and $Y(y,r)$, and we have

$$\begin{aligned}\mathbb{E}[|V(Y_T(y,r)) + X_T(x,q)Y_T(y,r) - U(X_T(x,q))|] =\\ \mathbb{E}[V(Y_T(y,r)) + X_T(x,q)Y_T(y,r) - U(X_T(x,q))]\\ \leq v(y,r) + xy + qr - u(x,q) = 0,\end{aligned}$$

which implies (19) and (20).

To prove that $-X(x,q)$ is acceptable, observe that there exists a process $X \in \mathcal{X}(x,q)$ such that $-X$ is acceptable and $X_T \geq X_T(x,q)$ (see [Sio12a, Lemma 2]); since $X(x,q)$ is an optimizer, this implies $X_T = X_T(x,q)$. Then $X(x,q) - X \in \mathcal{X}(0,0)$, so Remark 4 gives the thesis.

An example that shows that the inclusions in item 2 can be strict is obtained taking U to be a power utility of exponent α and considering a market with no stocks and one contingent claim uniformly distributed in $[-1,1]$. Indeed one can compute explicitly $u(x,q)$ and then with the aid of [Roc70, Theorems 23.2 and 23.3] one can obtain that when α is between 1 and 0 one has $\mathcal{K} = dom(u)$, when α is between 0 and -1 one has $\mathcal{K} = dom(\partial u)$ and $\bar{\mathcal{K}} \setminus \{0\} = dom(u)$, and when α is smaller than -1 one has $\bar{\mathcal{K}} \setminus \{0\} = dom(\partial u)$ and $\bar{\mathcal{K}} = dom(u)$.

To conclude the proof of item 3, we only need to prove that $\partial u(\bar{\mathcal{K}}) \subseteq \mathcal{L}$, since item 1 implies that $\partial u(\mathbb{R}^n) = dom(\partial v)$ and $\partial v(\mathbb{R}^n) = dom(\partial u)$ (see [Roc70, Theorem 23.5]); the result on the partial derivatives of v follows then from [Roc70, Theorem 23.3]. Notice first that

$$\partial u(\bar{\mathcal{K}}) = dom(\partial v) \subseteq \bar{\mathcal{L}}.$$

Now $\partial u(\bar{\mathcal{K}}) \subseteq \mathcal{L}$ follows exactly as in the proof of [HK04, Theorem 2] once we notice that the following relationship holds for every $(x,q) \in \bar{\mathcal{K}}$ (and not just for $(x,q) \in \mathcal{K}$): for any non-negative measurable function g,

$$g \in \mathcal{C}(x,q) \iff \mathbb{E}[gh] \leq 1 \quad \forall h \in \tilde{\mathcal{D}}(x,q), \tag{21}$$

where by definition

$$\begin{aligned}A(x,q) &:= \{(y,r) \in \bar{\mathcal{L}} : xy + qr \leq 1\},\\ \tilde{\mathcal{D}}(x,q) &:= \textstyle\bigcup_{(y,r)\in A(x,q)} \mathcal{D}(y,r).\end{aligned}$$

The equivalence (21) follows from [HK04, Proposition 1], which still holds (with the same proof) if $\mathcal{K}$ and $\mathcal{L}$ are replaced with $\bar{\mathcal{K}}$ and $\bar{\mathcal{L}}$[7]. □

We remark that there is a simple general condition which is equivalent to the domain of u being the whole of $\bar{\mathcal{K}}$, that is, $U(0) := \lim_{x\to 0+} U(x)$ needs to be real valued. Indeed trivially

$$u(x,q) \geq u(0,0) \text{ at all } \quad (x,q) \in \bar{\mathcal{K}},$$

so $dom(u) = \bar{\mathcal{K}}$ iff $u(0,0) \in \mathbb{R}$, and then Remark 4 implies that $u(0,0) = U(0)$.

[7]This is true also for [HK04, Lemma 10], on which [HK04, Proposition 1] relies

7 Relation between the two optimization problems

PROOF OF THEOREM 2. Item 2 is part of item 4 in Theorem 6, and item 3 is trivial. For the proof of item 1, fix x and p, and let f be the function $f(q) := -u(x-pq,q)$, which is proper convex and lower semi-continuous (by Theorem 6); its minimizer is then given by $\partial f^*(0)$ (see [HUL01, Formula (1.4.6), Chapter E]), where f^* is the Fenchel-Legendre transform of f. To compute f^* we write $f = g \circ A$ with $g(a,b) := -u(-(a,b))$, $A(q) := (-x+pq,-q)$, and apply [HUL01, Chapter E, Theorem 2.2.3] to find

$$f^*(\mu) = min_{(y,r)}\{v(y,r)+xy : yp-r=\mu\}.$$

The previous expression for f^* and allow us to compute $\partial f^*(0)$ using [HUL01, Chapter D, Theorem 4.5.1], and to find

$$\partial f^*(0) = \{q : (x-pq,q) \in -\partial v(y,r)\}, \tag{22}$$

where (y,r) is any solution of

$$min_{(y,r)}\{v(y,r)+xy : yp-r=0\}. \tag{23}$$

Item 1 of Theorem 1 then implies that problem (23) has as unique solution, namely $(\tilde{y},\tilde{y}p)$, with $\tilde{y} = \tilde{u}'(x)$. Stitching the pieces together, we obtain that the function $q \mapsto u(x-pq,q)$ is maximized at the points $(x-p\tilde{q},\tilde{q})$ in $-\partial v(\tilde{u}'(x),\tilde{u}'(x)p)$, which concludes the proof of item 1.

The equivalence of items 4c and 4d is part of [HK04, Lemma 3], and the the equivalence of items 4a and 4b follows from item 3. If item 4d holds, item 1 shows that $-\nabla v(\tilde{y},\tilde{y}p)$ is the unique solution to (7); in particular item 4b holds. To show that item 4b implies item 4d we have to proceed differently[8]; so, let us assume that item item 4c does not hold, and let $q' \neq 0$ be such that $q'f$ is replicated by a process $X' \in \mathcal{X}(x')$. This implies that $x' = q'p$, since p is an arbitrage-free price. Now, let $(\tilde{X},\tilde{q})$ be an solution to (4); then, $(\tilde{X}+x'-X',\tilde{q}+q')$ is also a solution to (4) (since it has the same final payoff), so item 4a does not hold. This concludes the proof of item 4. □

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References

[AIS05] M. Jonsson, A. Ilhan and R. Sircar. Optimal investment with derivative securities. *Finance Stoch.*, 9(4):585–595, 2005.

[BS99] W. Brannath and W. Schachermayer. A bipolar theorem for $L^0_+(\Omega,\mathcal{F},P)$. In *Séminaire de Probabilités, XXXIII*, volume 1709 of *Lecture Notes in Math.*, pages 349–354. Springer, Berlin, 1999.

[8] Since item 4b implies that $-\nabla v(\tilde{y},\tilde{y}p)$ exists at all $\tilde{y} > 0$, but only for the one fixed p which we used in problem (7), and not for all p such that $(1,p) \in \mathcal{L}$.

[DS94] F. Delbaen and W. Schachermayer. A general version of the fundamental theorem of asset pricing. *Math. Ann.*, 300(3):463–520, 1994.

[DS95] F. Delbaen and W. Schachermayer. The no-arbitrage property under a change of numéraire. *Stochastics Stochastics Rep.*, 53(3-4):213–226, 1995.

[DS97] F. Delbaen and W. Schachermayer. The Banach space of workable contingent claims in arbitrage theory. *Ann. Inst. H. Poincaré Probab. Statist.*, 33(1):113–144, 1997.

[DS98] F. Delbaen and W. Schachermayer. The fundamental theorem of asset pricing for unbounded stochastic processes. *Math. Ann.*, 312(2):215–250, 1998.

[HK79] M. Harrison and D. Kreps. Martingales and arbitrage in multiperiod security markets. *Journal of Economic Theory*, 20:381–408, 1979.

[HK04] J. Hugonnier and D. Kramkov. Optimal investment with random endowments in incomplete markets. *Ann. Appl. Probab.*, 14(2):845–864, 2004.

[HP81] M. Harrison and S. Pliska. Martingales and stochastic integrals in the theory of continuous trading. *Stochastic Process. Appl.*, 11(3):215–260, 1981.

[HUL01] J. Hiriart-Urruty and C. Lemaréchal. *Fundamentals of convex analysis.* Grundlehren Text Editions. Springer-Verlag, Berlin, 2001.

[IKX91] S. Shreve, I. Karatzas, J. Lehoczky and G. Xu. Martingale and duality methods for utility maximization in an incomplete market. *SIAM J. Control Optim.*, 29(3):702–730, 1991.

[JCW01] W. Schachermayer, J. Cvitanić and H. Wang. Utility maximization in incomplete markets with random endowment. *Finance Stoch.*, 5(2):259–272, 2001.

[KS99] D. Kramkov and W. Schachermayer. The asymptotic elasticity of utility functions and optimal investment in incomplete markets. *Ann. Appl. Probab.*, 9(3):904–950, 1999.

[KS03] D. Kramkov and W. Schachermayer. Necessary and sufficient conditions in the problem of optimal investment in incomplete markets. *Ann. Appl. Probab.*, 13(4):1504–1516, 2003.

[KS06] D. Kramkov and M. Sîrbu. Sensitivity analysis of utility-based prices and risk-tolerance wealth processes. *Ann. Appl. Probab.*, 16(4):2140–2194, 2006.

[LZ] K. Larsen and G. Zitkovic. Utility maximization under convex constraints. *Forthcoming in Ann. Appl. Probab.*

[Mer69] R. Merton. Lifetime portfolio selection under uncertainty: the continuous-time case. *Rev. Econom. Statist.*, pages 247–257, 1969.

[Mer71] R. Merton. Optimum consumption and portfolio rules in a continuous-time model. *J. Econom. Theory*, 3(4):373–413, 1971.

[Roc70] T. Rockafellar. *Convex analysis.* Princeton Mathematical Series, No. 28. Princeton University Press, Princeton, N.J., 1970.

[Ros76] S. Ross. The arbitrage theory of capital asset pricing. *Journal of Economic Theory*, 13:341–360, 1976.

[Sio12a] P. Siorpaes. Do arbitrage-free prices come from utility maximization? *Preprint*, 2012.

[Sio12b] P. Siorpaes. Price dependence in the problem of optimal investment. *Preprint*, 2012.